Precise moment asymptotics for the stochastic parabolic Anderson model of a time-derivative Gaussian noise

Heyu Li

(Joint work with Prof. Xia Chen)

School of Mathematics, Jilin University

2019.07.13

Outline

1. Introduction
2. Main Results
3. Sketch of the Proof
4. Conclusion and Further work

Outline

1 Introduction
- Parabolic Anderson Model
- Framework

2 Main Results

3 Sketch of the Proof

4 Conclusion and Further work

Parabolic Anderson Model

Classical parabolic Anderson model

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \kappa \Delta u(t,x) + u(t,x)V(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases}$$

where $\kappa > 0$ is diffusion coefficient, Δ is Laplace operator, $V(t,x)$ is random field.

Parabolic Anderson model has an interpretation as a population growth model. In addition, it has also provided a model for a polymer in random media. Further, it is closely related to equations for many other models, for example, the Stepping Stone model, Catalytic branching, Burger's equation, and the KPZ equation.

Intermittency property

Intermittency

For any $p \geq 1$, define following moment Lyapunov exponent,

$$\gamma(p) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u(t,x)|^p.$$

If $\gamma(p)/p$ is strictly increasing, i.e.

$$\gamma(1) < \frac{\gamma(2)}{2} < \cdots < \frac{\gamma(p)}{p} < \cdots,$$

then we call random field $u(t,x)$ is full intermittency. For all $p \geq 2$, if we have $\gamma(2) > 0$ and $\gamma(p) < \infty$, then we say that $u(t,x)$ is weakly intermittent.

Intermittency Problem

- CARMONA R A, MOLCHANOV S A. Parabolic Anderson problem and intermittency[J]. Memoirs of the American Mathematical Society, 1994, 518(518):125.
- CHEN X, HU Y Z, SONG J, et al. Exponential asymptotics for time-space Hamiltonians[J]. Annals de l'Institut Henri Poincaré-Probabilités et Statistiques, 2015, 51(4): 1529-1561.
- CHRN X, HU Y Z, SONG J, et al. Temporal asymptotics for fractional parabolic Anderson model[J]. Electronic Journal of Probability, 2018, 23(14): 1-39.

High Moment Asymptotics

- CHEN X. Spatial asymptotics for the parabolic Anderson models with generalized time-space Gaussian noise. The Annals of Probability, 2016, 44(2): 1535-1598.
- CHEN X. Moment asymptotics for parabolic Anderson equation with fractional time-space noise: in Skorokhod regime. Annals de l'Institut Henri Poincaré-Probabilités et Statistiques, 2017, 53(2): 819-841.

Our Model

We consider the following parabolic Anderson equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + u(t,x)\dfrac{\partial W}{\partial t}(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

with the Gaussian noise $\frac{\partial W}{\partial t}(t,x)$ that is formally given as the time derivative of the mean-zero Gaussian field $\{W(t,x);\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\}$ with the covariance function

$$\mathrm{Cov}\Big(W(t,x), W(s,y)\Big) = \frac{1}{2}(t^{2H_0} + s^{2H_0} - |t-s|^{2H_0})\Gamma(x,y),$$

where $(t,x),(s,y) \in \mathbb{R}^+ \times \mathbb{R}^d$, and the time Hurst parameter $H_0 \in (0,1)$.

Mathematically, $\frac{\partial}{\partial t}W(t,x)$ is defined as a generalized centered Gaussian field with

$$\mathrm{Cov}\left(\frac{\partial}{\partial t}W(t,x), \frac{\partial}{\partial s}W(s,y)\right) = \gamma_0(t-s)\Gamma(x,y), \qquad (t,x),(s,y) \in \mathbb{R}^+ \times \mathbb{R}^d.$$

Here

$$\gamma_0(t-s) = \begin{cases} H_0(2H_0-1)|t-s|^{-(2-2H_0)} & H_0 > 1/2, \\ \delta_0(t-s) & H_0 = 1/2, \end{cases}$$

when $H_0 < 1/2$,

$$\gamma_0(u) = \frac{\Gamma(2H_0+1)\sin(\pi H_0)}{2\pi}\int_{\mathbb{R}} e^{i\lambda u}|\lambda|^{1-2H_0}d\lambda, \qquad u \in \mathbb{R}.$$

Chen et al.(2018) proved that for the time Hurst parameter $H_0 \in (0, 1/2)$, if $\Gamma(x, y)$ satifies:

(H1) There exist some constants $\alpha \in (0, 1]$ and $C_0 > 0$ such that

$$\Gamma(x, x) + \Gamma(y, y) - 2\Gamma(x, y) \leq C_0|x - y|^{2\alpha}, \text{ for all } x \text{ and } y \in \mathbb{R}^d$$

and $2H_0 + \alpha > 1$, then the following Feynman-Kac formula:

$$u(t, x) = \mathbb{E}_x\left[u_0(B_t)\exp\left\{\int_0^t W(ds, B_{t-s})ds\right\}\right], \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.2}$$

is the solution to (1.1).

Where $\{B_t;\ t \geq 0\}$ is a d-dimensional Brownian motion starting from $x \in \mathbb{R}^d$, independent of $\{W(t,x);\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\}$, and "$\mathbb{E}_x$"stands for the Brownian expectation. The time integral $\int_0^t W(ds, B_{t-s})ds$ is defined by the approximation

$$\int_0^t W(ds, B_{t-s}) := \lim_{\epsilon \to 0^+} \int_0^t \frac{\partial W_\epsilon}{\partial s}(s, B_{t-s})ds, \quad \text{in } \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$$

where $W_\epsilon(s,x)$ is a properly smoothed version of $W(s,x)$

$$W_\epsilon(s,x) = (2\epsilon)^{-1}\big(W(s+\epsilon,x) - W(s-\epsilon,x)\big).$$

Moreover, they established that for some nonnegative constants $\underline{C}, \overline{C}$, if

(H2) There exist some constants $\beta \in (0,1]$ and $C_2 > 0$ such that for all $M > 0$,

$$\Gamma(x,y) \geq C_2 M^{2\beta}, \quad \text{for all } x, y \in \mathbb{R}^d \text{ with } x_i, y_i \geq M, i = 1, \cdots, d.$$

(H3) There exist a constant $C_1 > 0$ such that for all $M > 0$,

$$|\Gamma(x,y)| \leq C_1 (1+M)^{2\alpha}, \quad \text{for all } x, y \in \mathbb{R}^d \text{ with } |x|, |y| \geq M.$$

The solution to (1.1) satisfies the following moment bounds

$$\underline{C} \exp\left\{ \underline{C} m^{\frac{2-H}{1-H}} t^{\frac{2H_0+H}{1-H}} \right\} \leq \mathbb{E} u^m(t,x) \leq \overline{C} \exp\left\{ \overline{C} m^{\frac{2-H}{1-H}} t^{\frac{2H_0+H}{1-H}} \right\}. \tag{1.3}$$

Outline

1 Introduction

2 Main Results

3 Sketch of the Proof

4 Conclusion and Further work

For simplicity, we assume that bounded initial condition

$$0 < \inf_{x\in\mathbb{R}^d} u_0(x) \leq \sup_{x\in\mathbb{R}^d} u_0(x) < \infty.$$

We assume that the space covariance function $\Gamma(x,y)$ has the homogeneity in the sense that

$$\begin{cases} \Gamma(Cx,Cx) = |C|^{2H}\Gamma(x,x), \\ \Gamma(x,x)+\Gamma(y,y)-2\Gamma(x,y) = \Gamma(x-y,x-y). \end{cases} \tag{2.1}$$

for any $x, y \in \mathbb{R}^d$ and $C \in \mathbb{R}$, where the constant $H \in (0,1)$.

Theorem (H.Y. Li, X. Chen'19)

Assume that $0 < H_0, H < 1$ and $1 - 2H_0 < H < 1$, if $\Gamma(x, y)$ is locally bounded and holds condition (2.1), for every $x \in \mathbb{R}^d$, we have

$$\lim_{t \vee m \to \infty} t^{-\frac{2H_0+H}{1-H}} m^{-\frac{2-H}{1-H}} \log \mathbb{E} u^m(t,x) = \mathcal{E}(H_0). \tag{2.2}$$

Let $C_0\left\{[0,1], \mathbb{R}^d\right\}$ be the space of continuous functions $x(s)$: $[0,1] \longrightarrow \mathbb{R}^d$ with $x(0) = 0$ and $\mathcal{H}_d$ be the Cameron-Martin space given as

$$\mathcal{H}_d = \Bigg\{ x(\cdot) \in C_0\left\{[0,1], \mathbb{R}^d\right\};$$

$$x(s) \text{ is absolutely continuous and } \int_0^1 |\dot{x}(s)|^2 ds < \infty \Bigg\}.$$

- When $1/2 < H_0 < 1$,

$$\mathcal{E}(H_0) = \sup_{x \in \mathcal{H}_d} \left\{ \frac{C_{H_0}}{2} \int_0^1 \int_0^1 \frac{\Gamma(x(s), x(r))}{|s-r|^{(2-2H_0)}} dsdr - \frac{1}{2} \int_0^1 |\dot{x}(s)|^2 ds \right\}$$

- When $H_0 = 1/2$,

$$\mathcal{E}\left(\frac{1}{2}\right) = \sup_{x \in \mathcal{H}_d} \left\{ \frac{1}{2} \int_0^1 \Gamma(x(s), x(s)) ds - \frac{1}{2} \int_0^1 |\dot{x}(s)|^2 ds \right\}$$

- When $0 < H_0 < 1/2$,

$$\mathcal{E}(H_0) = \sup_{x \in \mathcal{H}_d} \left\{ \frac{|C_{H_0}|}{4} \int_0^1 \int_0^1 \frac{\Gamma(x(s)-x(r), x(s)-x(r))}{|s-r|^{(2-2H_0)}} dsdr \right.$$
$$\left. + \frac{H_0}{2} \int_0^1 \left\{ s^{-(1-2H_0)} + (1-s)^{-(1-2H_0)} \right\} \Gamma(x(s), x(s)) ds - \frac{1}{2} \int_0^1 |\dot{x}(s)|^2 ds \right\}$$

where $C_{H_0} = H_0(2H_0 - 1)$.

In connection to the Feynman-Kac representation (1.2) and in view of the variance identities given in(3.11), (3.12) and (3.13) below, all variations can be unified into the following form:

$$\mathcal{E}(H_0) = \sup_{x \in \mathcal{H}_d} \left\{ \frac{1}{2} \mathrm{Var}\left(\int_0^1 W\big(ds, x(s)\big) \right) - \frac{1}{2} \int_0^1 |\dot{x}(s)|^2 ds \right\}.$$

Clearly, $\mathcal{E}(H_0) > 0$ for every $0 < H_0 < 1$. In addition, we can show that $\mathcal{E}(H_0) < \infty$ whenever $1 - 2H_0 < H < 1$.

Remark

When $W(t,x)$ is a fractional Brownian sheet with the Hurst parameter $(H_0, H_1, \cdots, H_d)$,

$$\Gamma(x,y) = \prod_{j=1}^{d} R_{H_j}(x_j, y_j), \qquad x = (x_1, \cdots, x_d),\ y = (y_1, \cdots, y_d) \in \mathbb{R}^d,$$

where

$$R_{H_j}(x_j, y_j) = \frac{1}{2}\left\{|x_j|^{2H_j} + |y_j|^{2H_j} - |x_j - y_j|^{2H_j}\right\}, \qquad j = 1, \cdots, d.$$

One can verify the homogeneous assumption (2.1) with $H = H_1 + \cdots + H_d$.

Remark

When $W(t,x)$ is a spatial radial fractional Brownian sheet with Hurst parameter (H_0, H),

$$\Gamma(x,y) = \frac{1}{2}\{|x|^{2H} + |y|^{2H} - |x-y|^{2H}\}, \quad x, y \in \mathbb{R}^d,$$

also holds the homogeneous condition (2.1).

Corollary

When $\frac{\partial W}{\partial t}(t,x)$ is one-dimension fractional Brownian motion with Hurst parameter H in space and white in time, the evaluation of the variation $\mathcal{E}(H_0)$ in Theorem is

$$\begin{aligned}\mathcal{E}(1/2) &= \sup_{x\in\mathcal{H}_1}\left\{\frac{1}{2}\int_0^1 |x(s)|^{2H}ds - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 ds\right\} \\ &= 2^{\frac{1+H}{1-H}}(1-H^2)H^{\frac{2H}{1-H}}B\left(\frac{1}{2},\frac{1}{2H}\right)^{-\frac{2H}{1-H}}.\end{aligned}$$

where $B(\cdot,\cdot)$ is the beta function. We proved the Corollary by using a historic conclusion by Strassen(1964) and Lagrange multipliers.

In addition, we assume that $\Gamma(x,y)$ satisfies the following asymptotic homogeneous condition,

$$\begin{cases} \Gamma(x,x) \sim \Gamma_h(x,x), \qquad (|x| \to +\infty), \\ \Gamma(x,x) + \Gamma(y,y) - 2\Gamma(x,y) = \Gamma(x-y, x-y), \end{cases} \tag{2.3}$$

where $\Gamma_h(x,x)$ is the space covariance function holds homogeneity (2.1). Then we have the same conclusion as before,

$$\lim_{t \vee m \to \infty} t^{-\frac{2H_0+H}{1-H}} m^{-\frac{2-H}{1-H}} \log \mathbb{E} u^m(t,x) = \mathcal{E}(H_0).$$

- When $1/2 < H_0 < 1$,

$$\mathcal{E}(H_0) = \sup_{x \in \mathcal{H}_d} \left\{ \frac{C_{H_0}}{2} \int_0^1 \int_0^1 \frac{\Gamma_h(x(s), x(r))}{|s-r|^{(2-2H_0)}} ds dr - \frac{1}{2} \int_0^1 |\dot{x}(s)|^2 ds \right\};$$

- when $H_0 = 1/2$,

$$\mathcal{E}\left(\frac{1}{2}\right) = \sup_{x \in \mathcal{H}_d} \left\{ \frac{1}{2} \int_0^1 \Gamma_h(x(s), x(s)) ds - \frac{1}{2} \int_0^1 |\dot{x}(s)|^2 ds \right\};$$

- when $0 < H_0 < 1/2$,

$$\mathcal{E}(H_0) = \sup_{x \in \mathcal{H}_d} \left\{ \frac{|C_{H_0}|}{4} \int_0^1 \int_0^1 \frac{\Gamma_h(x(s)-x(r), x(s)-x(r))}{|s-r|^{(2-2H_0)}} ds dr \right.$$
$$\left. + \frac{H_0}{2} \int_0^1 \left\{ s^{-(1-2H_0)} + (1-s)^{-(1-2H_0)} \right\} \Gamma_h(x(s), x(s)) ds - \frac{1}{2} \int_0^1 |\dot{x}(s)|^2 ds \right\}.$$

Outline

1 Introduction

2 Main Results

3 Sketch of the Proof

3.1 Asymptotics for $\mathbb{E}u^m(t,0)$

3.2 Asymptotics for $\mathbb{E}u^m(t,x)$

4 Conclusion and Further work

Firstly, we prove the Theorem in the case when $x = 0$, that is,

$$\lim_{t\vee m\to\infty} t^{-\frac{2H_0+H}{1-H}} m^{-\frac{2-H}{1-H}} \log \mathbb{E}u^m(t,0) = \mathcal{E}(H_0). \tag{3.1}$$

Recall the Feynman-Kac representation

$$u(t,x) = \mathbb{E}_x\left[u_0(B_t)\exp\left\{\int_0^t W(ds, B_{t-s})ds\right\}\right], \tag{3.2}$$

we found that the solution $u(t,x)$ is monotonic in the initial state $u_0(x)$. By the bounded initial condition, therefore, we may assume $u_0(x) = 1$.

Taking expectation with respect to W, by Fubini theorem and Gaussian property,

$$\mathbb{E}u(t,x) = \mathbb{E}_x \exp\left\{\frac{1}{2}\mathrm{Var}\left(\int_0^t W(ds, B_{t-s})\Big|B\right)\right\},$$

where $\mathrm{Var}\,(\cdot|B)$ is the variance conditioning on the Brownian motion B_t.

- When $H_0 > 1/2$,

$$\mathrm{Var}\left(\int_0^t W(ds, B_{t-s})\Big|B\right) = C_{H_0}\int_0^t\int_0^t |s-r|^{-(2-2H_0)}\Gamma(B_s, B_r)dsdr. \tag{3.3}$$

- When $H_0 = 1/2$,

$$\mathrm{Var}\left(\int_0^t W(ds, B_{t-s})\Big|B\right) = \int_0^t \Gamma(B_s, B_s)ds. \tag{3.4}$$

Lemma [L. Chen, Y. Z. Hu et al.'18]

Assume that $\Gamma(\cdot,\cdot)$ satisfies condition **(H1)**. Then for all $0 < t \le T$ and $\phi,\psi \in C^{H_0}([0,T])$ with $2H_0(2H_0-1)+H > 1/2$, the stochastic integral $I(\phi) := \int_0^t W(ds,\phi_s)$ exists and

$$\mathbb{E}\big[I(\phi)I(\psi)\big] = H_0\int_0^t s^{2H_0-1}\big[\Gamma(\phi_s,\psi_s)+\Gamma(\phi_{1-s},\psi_{1-s})\big]ds + \frac{2H_0(2H_0-1)}{2}\int_0^t\int_0^t |s-r|^{2H_0-2}\hat{\Gamma}\big(s,r,\phi_s,\psi_r\big)dsdr.$$

where

$$\hat{\Gamma}\big(s,r,\phi_s,\psi_r\big) = \frac{1}{2}\big[\Gamma(\phi_s,\psi_s)+\Gamma(\phi_r,\psi_r)-\Gamma(\phi_s,\psi_r)-\Gamma(\phi_r,\psi_s)\big].$$

- When $H_0 < 1/2$,

$$\begin{aligned}\mathrm{Var}&\left(\int_0^t W(ds, B_{t-s})\Big| B\right)\\ =&H_0\int_0^t s^{-(1-2H_0)}\big\{\Gamma(B_s,B_s)+\Gamma(B_{t-s},B_{t-s})\big\}ds \qquad (3.5)\\ &+\frac{|C_{H_0}|}{2}\int_0^t\int_0^t \frac{\Gamma(B_s-B_r,B_s-B_r)}{|s-r|^{2-2H_0}}dsdr.\end{aligned}$$

Set

$$t_m = m^{\frac{1}{2(1-H)}}t^{\frac{2H_0+H}{2(1-H)}}.$$

All the conditional variances satisfy the identity in law:

$$\mathrm{Var}\left(\int_0^t W(ds,B_{t-s})\Big|B\right)\overset{d}{=} m^{-1}t_m^2\mathrm{Var}\left(\int_0^1 W(ds,t_m^{-1}B_{1-s})\Big|B\right).$$

By Gaussian property,

$$u(t,0)=\mathbb{E}_0\exp\left\{m^{-1/2}t_m\int_0^1 W(ds,t_m^{-1}B_{1-s})\right\}.$$

Therefore,

$$\mathbb{E}u^m(t,0)=\mathbb{E}_0\exp\left\{\frac{1}{2}m^{-1}t_m^2\mathrm{Var}\left(\sum_{j=1}^m\int_0^1 W(ds,t_m^{-1}B^j_{1-s})\Big|B\right)\right\},$$

where $B_t^1,\cdots,B_t^m$ be independent d-dimensional Brownian motions with $B_0^j=0$ ($j=1,\cdots,m$), is the variance conditioning on $B_t^1,\cdots,B_t^m$, "$\mathbb{E}_0$" is the expectation with respect to the Brownian motions $B_t^1,\cdots,B_t^m$.

Schilder's theorem [Dembo and Zeitouni'98]

For the Brownian motion $B = \{B_s;\ s \in [0,1]\}$ which is viewed as a Gaussian random variable taking values in $C_0\left\{[0,1],\mathbb{R}^d\right\}$. Let the rate function $I(x)$ on $C_0\left\{[0,1],\mathbb{R}^d\right\}$ be defined as

$$I(x) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 ds & x \in \mathcal{H}_d \\ \infty & \text{elsewhere.} \end{cases}$$

then

$$\limsup_{\epsilon\to 0+} \epsilon^2 \log \mathbb{P}_0\{\epsilon B \in F\} \le -\inf_{x\in F} I(x) \qquad \forall \text{ every close set } F \text{ in } C_0\left\{[0,1],\mathbb{R}^d\right\},$$

$$\liminf_{\epsilon\to 0+} \epsilon^2 \log \mathbb{P}_0\{\epsilon B \in G\} \ge -\inf_{x\in G} I(x) \qquad \forall \text{ every open set } G \text{ in } C_0\left\{[0,1],\mathbb{R}^d\right\}.$$

Varadhan's integral lemma [Dembo and Zeitouni'98]

$\{Z\epsilon\}$ is a family of random variables taking values in the regular topological space $\mathcal{X}$, and $\{\mu_\epsilon\}$.denotes the probability measures associated with $\{Z\epsilon\}$. Suppose that $\{\mu_\epsilon\}$ satisfies the LDP with a good rate function $I : \mathcal{X} \to [0,\infty]$, and let $\Psi : \mathcal{X} \to \mathbb{R}$ be any continuous function. Assume further either the tail condition

$$\lim_{M\to\infty} \limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}\Big[\exp\Big\{\Psi(Z_\epsilon)/\epsilon\Big\}\mathbf{1}_{\{\Psi(Z_\epsilon)\geq M\}}\Big] = -\infty,$$

or the following moment condition for some $\theta > 1$,

$$\limsup_{\epsilon\to 0^+} \epsilon \log \mathbb{E}\Big[\exp\Big\{\theta\epsilon^{-1}\Psi(Z_\epsilon)\Big\}\Big] < \infty. \tag{3.6}$$

Then

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\exp\Big\{\epsilon^{-1}\Psi(Z_\epsilon)\Big\} = \sup_{x\in\mathcal{X}}\Big\{\Psi(x) - I(x)\Big\}. \tag{3.7}$$

Upper bound

By Jensen's inequality and independence of $B_t^1, \cdots, B_t^m$,

$$\mathbb{E}u^m(t,0) \le \left(\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2 \mathrm{Var}\left(\int_0^1 W(ds, t_m^{-1}B_{1-s})\Big|B\right)\right\}\right)^m. \tag{3.8}$$

Taking $t = 1$ and replacing B by $t_m^{-1}B$ in (3.3), (3.4) and (3.5), respectively, we have that

- When $H_0 > 1/2$,

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2 \mathrm{Var}\left(\int_0^1 W(ds, t_m^{-1}B_{1-s})\Big|B\right)\right\}\\ =&\mathbb{E}_0 \exp\left\{\frac{C_{H_0}}{2}t_m^2 \int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\left(t_m^{-1}B_s, t_m^{-1}B_r\right)dsdr\right\}.\end{aligned}$$

- When $H_0 = 1/2$,

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2 \mathrm{Var}\left(\int_0^1 W(ds, t_m^{-1}B_{1-s})\Big| B\right)\right\}\\ =&\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2 \int_0^1 \Gamma(t_m^{-1}B_s, t_m^{-1}B_s)ds\right\}.\end{aligned}$$

- When $H_0 < 1/2$,

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2 \mathrm{Var}\left(\int_0^1 W(ds, t_m^{-1}B_{1-s})\Big| B\right)\right\}\\ =&\mathbb{E}_0 \exp\left\{\frac{H_0}{2}t_m^2 \int_0^1 s^{-(1-2H_0)}\{\Gamma(t_m^{-1}B_s, t_m^{-1}B_s) + \Gamma(t_m^{-1}B_{1-s}, t_m^{-1}B_{1-s})\}ds\right.\\ &\left.+ \frac{|C_{H_0}|}{4}t_m^2 \int_0^1\int_0^1 \frac{\Gamma\left(t_m^{-1}(B_s - B_r), t_m^{-1}(B_s - B_r)\right)}{|s-r|^{2-2H_0}}dsdr\right\}.\end{aligned}$$

For $H_0 > 1/2$, let

$$\Psi(x) = \frac{C_{H_0}}{2}\int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(x(s),x(r)\big)dsdr, \qquad x \in C_0\Big\{[0,1],\mathbb{R}^d\Big\},$$

and for $H_0 = 1/2$, let

$$\Psi(x) = \frac{1}{2}\int_0^1 \Gamma\big(x(s),x(s)\big)ds, \qquad x \in C_0\Big\{[0,1],\mathbb{R}^d\Big\},$$

the functions are continuous on $C_0\Big\{[0,1],\mathbb{R}^d\Big\}$, and we can prove that they satisfy (3.7), then by (3.8), we have

$$\limsup_{t\vee m\to\infty} m^{-1}t_m^{-2}\log\mathbb{E}u^m(t,0) \le \mathcal{E}(H_0), \tag{3.9}$$

which is the desired upper bound for (3.1) in the case $H_0 \ge 1/2$.

To the case $H_0 < 1/2$, we set

$$\Psi(x) = \frac{H_0}{2}\int_0^1 s^{-(1-2H_0)}\Big\{\Gamma\big(x(s),x(s)\big)+\Gamma\big(x(1-s),x(1-s)\big)\Big\}ds + \frac{|C_{H_0}|}{4}\int_0^1\int_0^1 \frac{\Gamma\big(x(s)-x(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}dsdr.$$

The second part of the function is not continuous on $C_0\Big\{[0,1],\mathbb{R}^d\Big\}$. Given a small number $0<\delta<1$, set

$$D_\delta = \{(s,r)\in[0,1]^2;\ |s-r|\le\delta\},\qquad \hat{D}_\delta = [0,1]^2\setminus D_\delta.$$

Let

$$\Psi_1(x) = \frac{H_0}{2}\int_0^1 s^{-(1-2H_0)}\left\{\Gamma\big(x(s),x(s)\big)+\Gamma\big(x(1-s),x(1-s)\big)\right\}ds + \frac{|C_{H_0}|}{4}\iint_{\hat{D}_\delta}\frac{\Gamma\big(x(s)-x(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}dsdr,$$

$$\Psi_2(x) = \frac{|C_{H_0}|}{4}\iint_{D_\delta}\frac{\Gamma\big(x(s)-x(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}dsdr.$$

By Hölder's inequality,

$$\mathbb{E}_0\exp\left\{t_m^2\Psi(t_m^{-1}B)\right\} \le \left(\mathbb{E}_0\exp\left\{t_m^2 p\Psi_1(t_m^{-1}B)\right\}\right)^{1/p}\left(\mathbb{E}_0\exp\left\{t_m^2 q\Psi_2(t_m^{-1}B)\right\}\right)^{1/q}. \tag{3.10}$$

By (3.8),

$$\lim_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\left\{t_m^2 p\Psi_1(t_m^{-1}B)\right\} = p^{\frac{1}{1-H}}\mathcal{E}(H_0),$$

By the assumption $1-2H_0<H<1$, we have

$$\limsup_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\left\{t_m^2 q\Psi_2(t_m^{-1}B)\right\} \le C_q\delta^{\frac{\alpha}{1-H}},$$

where $\alpha>0$, the constant $C_q>0$ is independent of δ.

Let $\delta\to 0^+$ and then $p\to 1^+$,

$$\limsup_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\left\{\frac{1}{2}t_m^2\mathrm{Var}\left(\int_0^1 W(ds, t_m^{-1}B_{1-s})\Big|B\right)\right\}\le\mathcal{E}(H_0).$$

Therefore, the desired upper bound (3.9) follows from (3.6) in the setting $H_0<1/2$.

Lower bound

Recall the moment representation

$$\mathbb{E}u^m(t,0) = \mathbb{E}_0 \exp\left\{\frac{1}{2} m^{-1} t_m^2 \mathrm{Var}\left(\sum_{j=1}^m \int_0^1 W(ds, t_m^{-1} B_{1-s}^j)\Big| B\right)\right\},$$

For any $x \in C_0\left\{[0,1];\mathbb{R}^d\right\}$, define the W-measurable random variable

$$\eta(x) := \int_0^1 W\big(ds, x(1-s)\big).$$

Let $y \in \mathcal{H}_d$ be fixed but arbitrary, we have

$$\mathrm{Var}\left(\frac{1}{m}\sum_{j=1}^m \eta(t_m^{-1}B^j)\Big|B\right) \geq -\mathrm{Var}\,(\eta(y)) + \frac{2}{m}\sum_{j=1}^m \mathrm{Cov}\,\left(\eta(y), \eta(t_m^{-1}B^j)\big|B\right).$$

By independence of $B_t^j, j = 1, \cdots, m$,

$$\mathbb{E}u^m(t,0) \geq \exp\Big\{ -\frac{1}{2} m t_m^2 \mathrm{Var}\,(\eta(y))\Big\}\Big(\mathbb{E}_0 \exp\Big\{ t_m^2 \mathrm{Cov}\,(\eta(y), \eta(t_m^{-1}B)\big|B)\Big\}\Big)^m,$$

In addition, we can claim that for all $0 < H_0 < 1$,

$$\liminf_{t\vee m\to\infty} t_m^{-2} \log \mathbb{E}_0 \exp\Big\{ t_m^2 \mathrm{Cov}\,\big(\eta(y), \eta(t_m^{-1}B)\big|B\big)\Big\}$$
$$\geq \sup_{x\in\mathcal{H}_d}\Big\{ \mathrm{Cov}\,\big(\eta(y), \eta(x)\big) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 ds\Big\}.$$

Picking $x = y$,

$$\liminf_{t\vee m\to\infty} m^{-1}t_m^{-2}\log \mathbb{E}u^m(t,0) \geq \frac{1}{2}\mathrm{Var}\left(\eta(y)\right) - \frac{1}{2}\int_0^1 |\dot{y}(s)|^2 ds,$$

Because $y \in \mathcal{H}_d$ can be arbitrary, taking supremum over y leads to

$$\liminf_{t\vee m\to\infty} m^{-1}t_m^{-2}\log \mathbb{E}u^m(t,0) \geq \sup_{x\in\mathcal{H}_d}\left\{\frac{1}{2}\mathrm{Var}\left(\eta(x)\right) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 ds\right\},$$

Finally, the desired lower bound follows from the variance representation

- when $H_0 > 1/2$,

$$\mathrm{Var}\left(\eta(x)\right) = C_{H_0}\int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(x(s),x(r)\big)dsdr; \tag{3.11}$$

- when $H_0 = 1/2$,

$$\mathrm{Var}\left(\eta(x)\right) = \int_0^1 \Gamma\big(x(s),x(s)\big)ds; \tag{3.12}$$

- when $H_0 < 1/2$,

$$\mathrm{Var}\left(\eta(x)\right) = H_0\int_0^1 s^{-(1-2H_0)}\big\{\Gamma(x(s),x(s))+\Gamma(x(1-s),x(1-s))\big\}ds \tag{3.13}$$
$$+\frac{|C_{H_0}|}{2}\int_0^1\int_0^1 \frac{\Gamma\big(x(s)-x(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}dsdr.$$

For $u_0(x) = 1$, (1.2) can be rewritten as

$$u(t,x) = \mathbb{E}_0 \exp\left\{ \int_0^t W(ds, x + B_{t-s}) \right\}.$$

Given $\theta > 0$, write $u_\theta(t,x)$ for the solution to (1.1) with the constant 1 as its initial value and with $W(t,x)$ being replaced by $\theta W(t,x)$. By the Hölder's inequality again,

$$\begin{aligned} \mathbb{E}u(t,x)^m &\le \Big(\mathbb{E}u_p^m(t,0)\Big)^{1/p} \\ &\times \left(\mathbb{E}_0 \exp\left\{\frac{q^2}{2} m \mathrm{Var}\left(\int_0^t W(ds, x+B_{t-s}) - \int_0^t W(ds, B_{t-s}) \Big| B\right)\right\}\right)^{m/q}. \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}u(t,x)^m &\ge \Big(\mathbb{E}u_{1/p}^m(t,0)\Big)^{p} \\ &\times \left(\mathbb{E}_0 \exp\left\{\frac{q^2}{2p} m \mathrm{Var}\left(\int_0^t W(ds, x+B_{t-s}) - \int_0^t W(ds, B_{t-s}) \Big| B\right)\right\}\right)^{-\frac{pm}{q}}. \end{aligned}$$

we proved that

$$\mathrm{Var}\left(\int_0^t W(ds, x+B_{t-s}) - \int_0^t W(ds, B_{t-s}) \Big| B\right) = \Gamma(x,x)t^{2H_0},$$

then

$$\begin{aligned}
&\exp\left\{-\frac{q}{2p}\Gamma(x,x)m^2t^{2H_0}\right\}\left(\mathbb{E}u^m_{1/p}(t,0)\right)^p \\
\leq &\mathbb{E}u^m(t,x) \\
\leq &\exp\left\{\frac{q}{2}\Gamma(x,x)m^2t^{2H_0}\right\}\left\{\mathbb{E}u^m_p(t,0)\right\}^{1/p}.
\end{aligned}$$

Replacing $u(t,0)$ by $u_{1/p}(t,0)$ and $u_p(t,0)$ in (3.1), respectively, we have

$$\lim_{t\vee m\to\infty} t^{-\frac{2H_0+H}{1-H}} m^{-\frac{2-H}{1-H}} \log \mathbb{E}u_{1/p}^m(t,0) = \mathcal{E}_{1/p}(H_0)$$

and

$$\lim_{t\vee m\to\infty} t^{-\frac{2H_0+H}{1-H}} m^{-\frac{2-H}{1-H}} \log \mathbb{E}u_p^m(t,0) = \mathcal{E}_p(H_0),$$

By the space homogeneity given in (2.1)

$$\mathcal{E}_p(H_0) = p^{\frac{2}{1-H}}\mathcal{E}(H_0), \qquad \mathcal{E}_{1/p}(H_0) = p^{-\frac{2}{1-H}}\mathcal{E}(H_0).$$

Letting $p \to 1^+$,

$$\lim_{t\vee m\to\infty} t^{-\frac{2H_0+H}{1-H}} m^{-\frac{2-H}{1-H}} \log \mathbb{E}u^m(t,x) = \mathcal{E}(H_0),$$

then we complete the proof of the Theorem for any $x \in \mathbb{R}^d$. □

Outline

1 Introduction

2 Main Results

3 Sketch of the Proof

4 Conclusion and Further work

Conclusion

We mainly consider precise moment asymptotics for the parabolic Anderson model of a time-derivative Gaussian noise.

- Firstly, we obtained the precise moment asymptotics for the equation with the Gaussian noise that is fractional in time and homogeneous in space.
- Secondly, we also considered the precise moment asymptotics for the model when the space covariance function condition is weaken to asymptotic homogeneity.

Further work

We find that there are still many problems that can be further discussed.

- To the parabolic Anderson model we concerned, can further consider the precise moment asymptotic for the equation in the case of space discreted.
- We can also study the fractional parabolic Anderson model with time-derivative Gaussian noise, consider the existence of Feynman-Kac representation for the solution and the property of the solution.

Thanks for your Attention!